An iterative algorithm for the determination of Voronoi vertices in polygonal and non-polygonal domains

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Abstract

We propose a new iterative algorithm for the computation of the vertices of a Voronoi diagram for a set of geometric objects of the euclidean plane. Each one of these vertices is the centre of the circle "touching" a triple of objects (passing through points or tangent to any other geometric object). The algorithm starts with an initial triple of points pertaining to each one of the three objects. It computes its circumcentre and the closest point (called foot) of each object from the circumcentre. These three feet form the starting triple for the next iteration. We geometrically demonstrate a necessary and sufficient condition for the general case. This iterative algorithm is used as a new method for constructing a dynamic Voronoi diagram for a set of points and straight line segments (see Gold and ***al.* *[4]).***

1 Introduction

The Voronoi diagram has many applications in a variety of disciplines, and has been widely treated in the literature (see Okabe and al. [5] and Aurenhammer [l] for a general survey). The Voronoi diagram has been introduced by the russian mathematician Georgii Fedeorovitch Voronoi in a treatise on quadratic forms theory (see Voronoi [7], [8]). The ordinary point Voronoi diagram is a partition of the plane, in the way that each object (point) partitions the euclidean plane into a region, that is the locus of points which are closer from that object than from any other object (see Preparata and Shamos [6]). The concept of Voronoi diagram has been extended in different kinds of generalizations: higher order Voronoi diagrams (extension of the set S of generators, see Preparata and Shamos [6]), weighted Voronoi diagrams (see Okabe and al. [5]), Voronoi diagrams with obstacles (see Shamos and Hoey [1975] in Preparata and Shamos [6]), Voronoi diagrams for areas, and Voronoi diagrams for lines. The line Voronoi diagram is a generalization of the ordinary point Voronoi diagram, by extending the set S to points, line segments, and any "geometric element consisting of line segments that are connected" (see Okabe and al. [5]). The line Voronoi diagram has been intensively studied by Drysdale [1979], Lee [1978], Lee and Drysdale [1981] (in Okabe [5]), and Kirkpatrick [1979] in Okabe [5]. It is possible to distinguish different kinds of line Voronoi diagrams (see Okabe [5]): Voronoi diagram for a set of points and straight line segments, Voronoi diagram for a set of circles, and Voronoi diagrams for a set of points, straight line segments and circular arcs.

The Voronoi diagram for a set of geometric objects of the euclidean plane is defined by the generalization of the ordinary point Voronoi diagram by extending the set of objects S to any geometric element. This partition of the plane forms a net, whose vertices are called Voronoi vertices, and whose edges are called Voronoi edges. Each Voronoi vertex is the common intersection of exactly three edges, and therefore each Voronoi vertex is equidistant from its three nearest objects. An iterative algorithm has been used for "hunting Voronoi vertices in non polygonal domains" (see Ferruci and al. [3]). In their algorithm, the exact shape description of the objects is not needed. The only assumption is "to be able to answer to queries of the form "given a point p and an object S, determine the closest point on S from p"" (Ferruci and al. [3]). Starting from a point p on the plane, they compute the closest point on each object. Then, they compute the circumcentre of these three points, that will be the point p for the next iteration. They have defined a necessary condition of convergence, based on the fact that the smallest circle containing three points and whose centre is inside the triangle formed by these three points is the circle circumscribed to the three points. The sufficient condition is that the next point p is inside the triangle formed by the closest point, of each one of the three objects from the previous point p.

2 Preliminaries

Let $\mathbb{N}$ be the set of integers, $\mathbb{R}$ be the set of reals: and $\mathbb{R}^2$ be the euclidean plane. Let P be a point of $\mathbb{R}^2$. and 0 be a geometric object, then let's define the distance from P to 0 as: $d(P,O) = inf \{d(P.M) / M \in O\}$.

Let $\mathcal{O}$ be the set of the n generators of the Voronoi diagram.

$\mathcal{V}(\mathcal{O}) = \bigcup_{i=1}^{n} \mathcal{V}(O_i) = \mathbb{R}^2$ where $\mathcal{V}(O_i) = \{ M \epsilon \mathbb{R}^2 / \forall j : d(M, 0;) \leq d(M,$ uj))} is the Voronoi region of the object 0,. Each Voronoi edge is a portion of bisector of two objects. These Voronoi edges intersect at points, called Voronoi vertices. Being the intersection of two bisectors, the Voronoi vertices are at the same distance from three objects.

Let H be the vectorial euclidean hyperplane corresponding to $\mathbb{R}^2$ in the oriented (see Berger [2]) three dimensional vectorial euclidean space E. Let $\overrightarrow{k}$ be the unitary vector of E normal to H.

Let O_1, O_2, and O_3 be three objects.

The iterative algorithm (see figure 1 page 3) starts with three arbitrary points (called feet) taken on each one of the three objects: F_{1_0}, F_{2_0}, and F_{3_0}. The centre C_0 of the circle $\mathcal{C}_0$ circumscribed to the triangle formed by these three feet is computed. Then, each one of the closest point of O_1, O_2, and O_3 from C_0: F_{1_1} ; F_{2_1}, and F_{3_1} is computed and used as the starting point (foot) for the next iteration. The iterations stop when the distance between the present centre and the last one is smaller than a user-defined tolerance.

Let $(F_{1_n})_{n \in \mathbb{N}}$, $(F_{2_n})_{n \in \mathbb{N}}$, and $(F_{3_n})_{n \in \mathbb{N}}$ be the sequences of the points (called feet) on each one of the three objects O_1, O_2, and O_3, closest to the centre of $\mathcal{C}_{n-1}$ except for n = 0 where the foot are arbitrary points on each one of the objects.

Let $\mathcal{C}_n$ be the circle passing through F_{1_n}, F_{2_n} , and F_{3_n} for $n \geq 0$.

Let (C_n) be the sequence of the centres of the circles $\mathcal{C}_n$ for $n \geq 0$.

Let $\mathcal{C}_{1_n}$ be the circle whose diameter is $[F_{1_n} C_n]$ for $n \geq 0$.

Let $\mathcal{C}_{2_n}$ be the circle whose diameter is $[F_{2_n} C_n]$ for $n \geq 0$.

Let $\mathcal{C}_{3_n}$ be the circle whose diameter is $[F_{3_n} C_n]$ for $n \geq 0$.

3 A necessary and sufficient condition of convergence

First let's suppose that there exists a Voronoi vertex v for the triple of objects (O_1, O_2, O_3). Then, the circle whose centre is v and whose radius is the euclidean distance from v to O_1 touches the three objects O_1, O_2, O_3 respectively at P, Q, R in the counterclockwise order ($P \in O_1$, $Q \in O_2, R \in O_3$). This implies that the three feet are in the anticlockwise order (R is on the left of $\overrightarrow{PQ}$ or equivalently: $\overrightarrow{PQ}$ x $\overrightarrow{PR}$. $\overrightarrow{k} \geq 0$) and $\overrightarrow{vQ}$ is between $\overrightarrow{vP}$ and $\overrightarrow{vR}$ (the oriented angles $\widehat{\overrightarrow{vP}\overrightarrow{vQ}}$ and $\widehat{\overrightarrow{vQ}\overrightarrow{vR}}$ are inferior to the oriented angle $\widehat{\overrightarrow{vP}\overrightarrow{vR}}$; see [2] for a survey of oriented angles).

Therefore, it is easy to see that the sequences of the feet $(F_{1_n})_{n \in \mathbb{N}}$, $(F_{2_n})_{n \in \mathbb{N}}$, and $(F_{3_n})_{n \in \mathbb{N}}$ should verify from some integer q, that the anticlockwise order of the feet is the expected one.

Now, let's suppose that we are at the iteration $n \geq q$ and the feet F_{1_n} , F_{2_n} , and F_{3_n} , are in the anticlockwise order.

We will consider now for each object O_i, the portion O_{i_n} of O_i inside the disk $\mathcal{D}_n$, whose boundary is $\mathcal{C}_n$. If O_i is a point, then $\forall i \in \mathbb{N} : O_{i_n} = O_i$. If $O_i \cap \mathcal{D}_n \neq \{ F_{i_n} \}$ then we will consider O_{i_n} open, and otherwise we will consider O_{i_n} closed. If Vi $\in \{1,2,3\} : O_i \cap \mathcal{D}_n = \{F_{i_n}\}$, then $\mathcal{C}_n$ is the circle touching the three Voronoi objects O_1, O_2, and O_3, at F_{1_n}, F_{2_n}, and F_{3_n} respectively. Its centre C_n is the Voronoi vertex corresponding to the triple of objects $\{O_1, O_2, O_3\}$. For each object O_i, any point of O_{i_n}, if it exists is closer from the centre of $\mathcal{C}_n$ than F_{i_n} and any other point of $O_i - O_{i_n}$.

Thus,

$$\forall i \in \{1,2,3\} : F_{i_{n+1}} \in O_{i_n}. \qquad (1)$$

If, and only if F_{1_n}, F_{2_n} , and F_{3_n} are in the anticlockwise order, $\overrightarrow{C_n F_{2_n}}$ is between $\overrightarrow{C_n F_{1_n}}$ and $\overrightarrow{C_n F_{3_n}}$ (the oriented angles $\widehat{\overrightarrow{C_n F_{1_n}}\overrightarrow{C_n F_{2_n}}}$ and $\widehat{\overrightarrow{C_n F_{2_n}}\overrightarrow{C_n F_{3_n}}}$ are inferior to the oriented angle $\widehat{\overrightarrow{C_n F_{1_n}}\overrightarrow{C_n F_{3_n}}}$; ***see figure 2*** page 3). Indeed, $\overrightarrow{C_n F_{1_n}}$, $\overrightarrow{C_n F_{2_n}}$, and $\overrightarrow{C_n F_{3_n}}$ are three radiuses of the circle passing through the three feet F_{1_n}, F_{2_n}, and F_{3_n}. In the counterclockwise order along that circle, F_{1_n} , F_{2_n} and F_{3_n} are in same order as their radiuses from C_n: $\overrightarrow{C_n F_{1_n}}$, $\overrightarrow{C_n F_{2_n}}$, and $\overrightarrow{C_n F_{3_n}}$.

For each object O_i, $\mathcal{C}_n \cap \mathcal{C}_{i_n} = \{ F_{i_n} \}$, and the common tangent of $\mathcal{C}_n$ and $\mathcal{C}_{i_n}$ is therefore the tangent of $\mathcal{C}_n$ at F_{i_n}. The edge orthogonal to the common tangent and passing through F_{i_n} is the edge

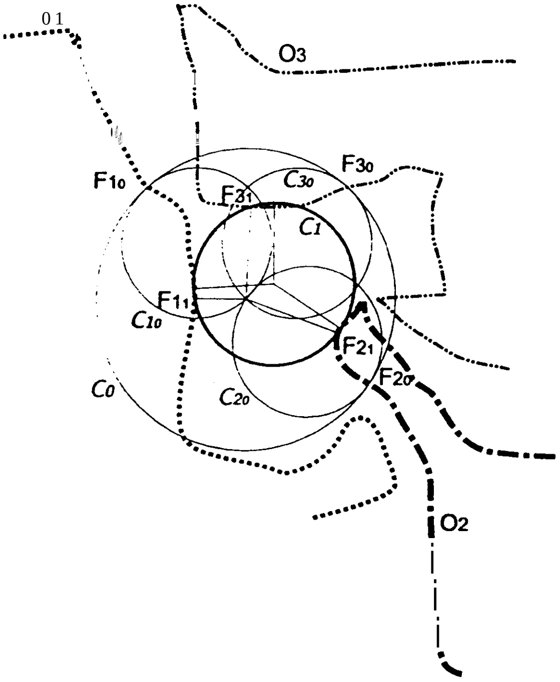


Figure 1: The iterative algorithm

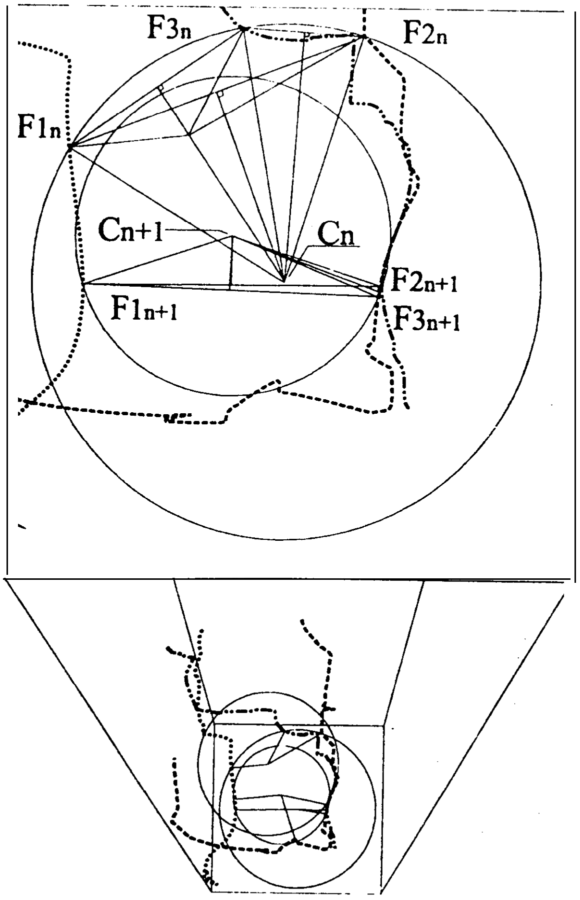


Figure 3: The convergence process when the order of the feet changes

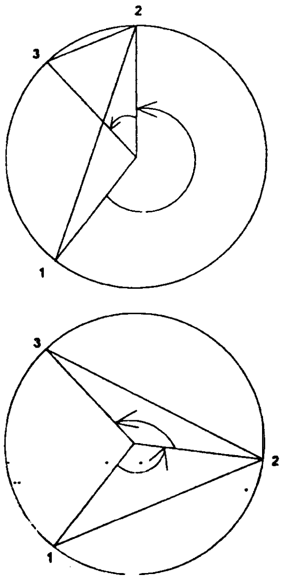


Figure 2: Visibility of three points from the circumcentre

$(F_{i_n}$ C,,). The vector $\overline{F_{i_n} C_n}$ gives the sense of the "movement" from F_{i_n} to $F_{i_{n+1}}$ along O_i, because it gives the relative position of O_{i_n} relatively to F_{i_n}.

As long as $\overrightarrow{F_{i_n}C_n} \cdot \overrightarrow{F_{i_{n+1}} C_{n+1}} \geq 0$, $F_{i_{n+2}}$ will be after F_{i_n} and $F_{i_{n+1}}$ along O_i. **NOW,** let **US** see the case where at least for one object $\overrightarrow{F_{i_n}C_n} \cdot \overrightarrow{F_{i_{n-1}}C_{n+1}} < 0$. We will prove that if all the objects verify $\overrightarrow{F_{i_n}C_n} \cdot \overrightarrow{F_{i_{n+1}}C_{n+1}} < 0$, then the order of the feet $F_{1_{n-1}}$; $F_{2_{n+1}}$, and $F_{3_{n-1}}$ has changed relatively to the order of the feet F_{1_n}, F_{2_n}, and F_{3_n} (see figure ***3*** page 3).

$\overrightarrow{F_{i_n}C_n} \cdot \overrightarrow{F_{i_{n+1}}C_{n+1}} < 0$ is equivalent to $\overrightarrow{C_nF_{i_n}} \cdot \overrightarrow{C_{n+1}F_{i_{n-1}}} < 0$. Let m_{ij_n} be the middle of $[F_{i_n} F_{j_n}]$.

Then, $\widehat{\overrightarrow{C_nF_{i_n}}\overrightarrow{C_nm_{ij_n}}} = \widehat{\overrightarrow{C_nm_{ij_n}}\overrightarrow{C_nF_{j_n}}} = \frac{1}{2}\widehat{\overrightarrow{C_nF_{i_n}}\overrightarrow{C_nF_{j_n}}} = \widehat{\overrightarrow{F_{k_n}F_{i_n}}\overrightarrow{F_{k_n}F_{j_n}}}$ where O_k is the third object. From $\overrightarrow{C_nF_{i_n}} \cdot \overrightarrow{C_{n+1}F_{i_{n+1}}} < 0$ and $\overrightarrow{C_nF_{j_n}} \cdot \overrightarrow{C_{n+1}F_{j_{n+1}}} < 0$ we get $\overrightarrow{C_nm_{ij_n}} \cdot \overrightarrow{C_{n+1}m_{ij_{n+1}}} < 0$ by passing to the bisectors..

Because of the fact that the circle centre C_n passing through $F_{i_{n+1}}$ is tangent to O_i, either $[F_{i_n} F_{j_n}]$ and $[F_{i_{n+1}} F_{j_{n+1}}]$ are not hidden by any of the two other objects (if there is a valid circle touching the three objects in that order, no object is hiding completely O_i from the third object) or they are both hidden by another object. Therefore the relative positions of $[F_{i_n} F_{j_n}]$ and $[F_{i_{n+1}} F_{j_{n+1}}]$ ***are*** the same, relatively to O_i and O_j, and we have:

$$\overrightarrow{m_{ij_n}F_{j_n}} \cdot \overrightarrow{m_{ij_{n+1}}F_{j_{n+1}}} = \frac{1}{4}\overrightarrow{F_{i_n}F_{j_n}} \cdot \overrightarrow{F_{i_{n+1}}F_{j_{n+1}}} \geq 0.$$

Therefore $\overrightarrow{C_nm_{ij_n}} \cdot \overrightarrow{C_{n+1} m_{ij_{n+1}}} < 0$ and $\overrightarrow{m_{ij_n}F_{j_n}} \cdot \overrightarrow{m_{ij_{n+1}}F_{j_{n+1}}} \geq 0$, and consequently $\left(\overrightarrow{C_nm_{ij_n}} \times \overrightarrow{C_nF_{j_n}}\right) \cdot \left(\overrightarrow{C_{n+1}m_{ij_{n+1}}} \times \overrightarrow{C_{n+1}F_{j_{n+1}}}\right) < 0$, and

$$\left(\overrightarrow{F_{k_n}F_{i_n}} \times \overrightarrow{F_{k_n}F_{j_n}}\right) \left(\overrightarrow{F_{k_{n+1}}F_{i_{n+1}}} \times \overrightarrow{F_{k_{n+1}}F_{j_{n+1}}}\right) < 0.$$

That means that without any loose of generality, we passed from the anticlockwise order of F_{k_n}, F_{l_n}, F_{m_n} to the anticlockwise order of $F_{k_{n+1}}$, $F_{m_{n+1}}$, F_{l_n}. The objects whose foot order has been changed (1 and m), either intersect between their two successive feet F_{l_n}, $F_{l_{n+1}}$, and F_{m_n}, $F_{m_{n+1}}$ respectively, or one of them (1 or m) hides the other one from the third object (k). However, we supposed at the beginning, that the order of the feet was the final order in which the searched circle has to touch each object.

Thus, the fact that there is at least one object for which $\overline{F_{i_n} C_n \cdot F_{i_{n+1}}C_{n+1}} > 0$, implies that $F_{i_{n+2}}$ ***can*** not be before F_{i_n} and $F_{i_{n+1}}$ along O_i, and therefore $F_{i_{n+2}}$ is either after F_{i_n} and $F_{i_{n+1}}$ or between F_{i_n} and $F_{i_{n+1}}$ along O_i. Therefore, if and only if, there exists a circle touching the three objects in that order, the sequences $(F_{i_n})_{n \in \mathbb{N}}$ will converge towards the closest points of $O_i, i \in \{1,2,3\}$ from the centre of that circle, and $(C_n)_{n \in \mathbb{N}}$ will converge towards the centre of that circle. We finally get the necessary and sufficient condition:

Theorem *1 The necessary and sufficient condition of convergence of this iterative algorithm is the following one:*

$$\exists p \in \mathbb{N} / \forall n \geq p : \overrightarrow{F_{1_n}F_{2_n}} \cdot \overrightarrow{F_{1_n}F_{3_n}} \geq 0 \qquad (2)$$

From this necessary and sufficient condition of convergence of this iterative algorithm, we get directly the initial conditions for this algorithm:

Lemma ***2 If we start from three feet in the expected final order: and $\forall$ (P, Q, R) $\in O_{1_0} \times O_{2_0} \times O_{3_0}$: $\overrightarrow{PQ} \times \overrightarrow{PR} \cdot \overrightarrow{k} > 0$. If, and only if, there exists a circle touching the three objects in the specified order, and for which there is no intersection with another object between the closest point of each object 0, from its centre and F_{i_0}, the sequences of $(F_{i_n})_{n \in \mathbb{N}}$ will converge towards the closest points of each one of the objects from the centre of that circle, and $(C_n)_{n \in \mathbb{N}}$ will converge towards the centre of that circle.***

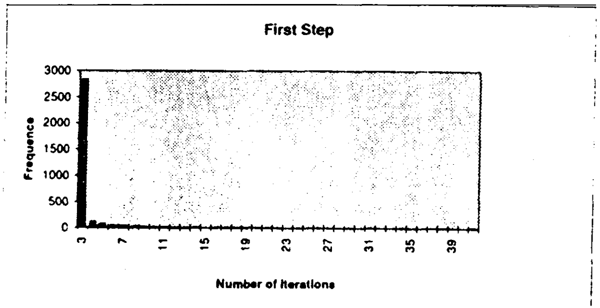
4 The Algorithm: description and statistical validity

An algorithm for the determination of Voronoi vertices for points and line segments has been developped using the precedent lemma. It is subdivided into three steps: two steps are necessary to satisfy the initial conditions, and the last step is the iterative algorithm itself.

In the first step, three starting feet in the expected order are chosen in order to satisfy to the first . Before trying to choose such feet, the extremities of the objects are checked to assess if it is possible. After, for each line segment object, a feet is randomly chosen till it is in the good order relatively to the other objects.

In the second step, the choice of the starting feet is corrected in order to satisfy the second initial condition: $\forall (P, Q, R) \in O_{1_0} \times O_{2_0} \times O_{3_0}$: $\overrightarrow{PQ} \times \overrightarrow{PR} \cdot \overrightarrow{k} > 0$.

In order to do so, each previously chosen foot at the iteration **n** is replaced by a feet for which the two extremities of $O_{i_{n+1}}$ are in the good order relatively to the other objects. Implementation of



Statistics	
Mean	4.620121951
Standard error	0.464252573
Standard deviation	26.56634109
Variance	706,939o817
Kurstosis	1207,749843
Asymmetrcoefficient	32.70444666
Extent	1132
Minimum	3
Maximum	1135
Sum	15610
Number of observations	3260
confidence value (95.0%)	0.910254764

Figure 4:

the above presented algorithm is done on points and line segments.

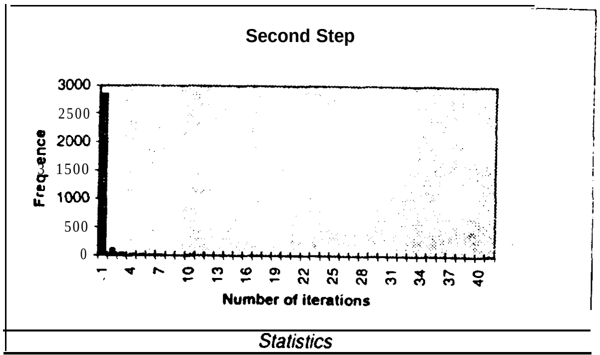
This algorithm has been implemented using Delphi on a Pentium based PC running Windows 95. Testing has been made using pseudo-random data including collisions. In order to generate special cases, we suppose that the second and third objects could be connected to the previous one.

The test data was composed of 10000 triples of objects. Among them, 3280 were assessed positively for circumcentre possibility. There was a valid circumcentre for 2997 cases, that is 91.37% of the previous set. The statistics for the 3280 cases mentioned above appear in the figures 4,5, and 6 page 5.

For the first step, we can see that 5 (4,82) iterations are needed in average. With a confidence interval of 95%, 6 (5,73) iterations are necessary.

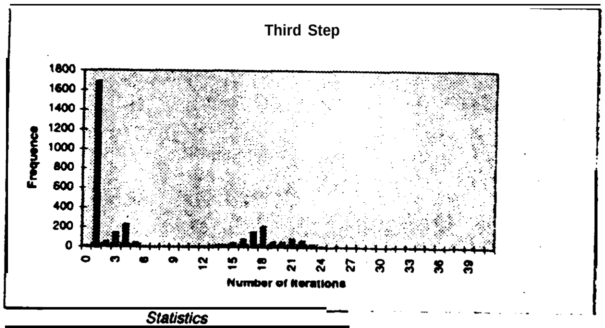
For the second step, we can see that 2 (1,44) iterations are needed in average. With a confidence interval of 95%, 2 (1,64)iterations are necessary.

Finally, for the third step, we can see that 7 (6,16) iterations are needed in average. With a confidence interval of 95%, 7 (6,44) iterations are necessary.



Statistics	
Mean	1,43818002
Standard error	0,104508027
Standard deviation	5,755537048
Variance	33.12620671
Kurstosis	292,9141068
Asymmetry coefficient	17,08998536
Extent	100
Minimum	
Maximum	101
Sum	4362
Number of observations	3033
Confidence value (95.0%)	0.204913883

Figure 5:



Statistics	
Mean	6.164164164
Standard error	0.136615491
Standard deviation	7,599434998
Variance	57.75141229
Kurstosis	-0.667693367
Asymmetry coefficient	1.04635822
Extent	27
Minimum	0
Maximum	27
Sum	16474
Number o observations	2997
Confidence value (95,0%)	0,27218343

Figure 6:

5 Conclusions

This algorithm is presently being applied to the routine, that computes the centre of the circle that touches three objects of a spatial data structure of points and oriented line segments. This routine is the fundamental part of a software of construction, and maintenance of a dynamic Voronoi spatial data structure for a set of points and oriented line segments. This spatial data structure is currently developed at the Industrial Chair of Geomatics ap plied to Forestry of the Centre for Research in Geomatics of Laval University, Quebec City (Canada) by Dr Christopher M. Gold.

Finally, this iterative algorithm is particularly interesting because the mathematical calculations involved in it (closest point and determinants) are directly transposable to the sphere.

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References

[l] F. Aurenhammer , Voronoi diagrams - A survey, (Institute for Information Processing, Technical University of Graz, Report 263)

[2] M. Berger, Géométrie, volume 2 : espaces euclidiens, triangles, cercles et spheres, (CEDIC/FERNAND NATHAN, Paris, 1979) 216p.

[3] V. Ferruci, M. Overmars, A. Rao, and J. VIeugels, Hunting Voronoi Vertices in Non-Polygonal Domains, (CCCG'94, Saskatoon, Canada, 1994) 45-50

[4] C.M. Gold, P.R. Remmele, and T. Roos, Voronoi Diagrams of Line Segments Made Easy, **(CCCG'95, Québec, Canada, 1995) 223-228**

[5]. A. Okabe, B. Boots, and K. Sugihara, Spatial **Tessellations, Concept. and Applications of Voronoi Diagrams, (John Wiley & Sons, 1992)** 532 **p.**

[6] F.P. Preparata, and M.I. Shamos, Computational Geometry, An Introduction, (Springer-Verlag, 1985) 398 p.

[7] G. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques, Premier Mémoire, Recherches sur les parallélloèdres primitifs, (Journal fur die Reine und Angewandte Mathematik, v. 134, 1908) 198-287

[8] G. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques, Deuxikme Mémoire, Recherches sur les **parallélloèdres** primitifs, (Journal fur die Reine und Angewandte Mathematik, v. 136, 1909) 67-181