Dynamic Additively Weighted Voronoi diagrams made easy

Francois Anton, Darka Mioc, and Christopher M. Gold
Industrial Chair of Geomatics, Université Laval
0722, Pavillon Casault, CRG
Sainte-Foy, Québec, Canada, G1K 7P4
E-mail: {francois,dmioc}@gmt.ulaval.ca & Christopher.Gold@scg.ulaval.ca

1. Introduction

The weighted Voronoi diagrams (multiplicatively, additively, compound, etc...) differ from the ordinary Voronoi diagram in the fact that the generators do not all have the same weight [6], [1]. The definition of these weighted Voronoi diagrams differs from the definition of the ordinary one in that the Euclidean distance is replaced by a weighted distance. In the case of the additively weighted Voronoi diagram, the weighted distance between a point and a generator is the Euclidean distance minus the weight of the generator. The additively weighted Voronoi diagram has been extensively studied by Ash and Bolker, under the name of hyperbolic Dirichlet tessellations (see section 2 of [3]). To date, only static algorithms have been developed for constructing the additively weighted Voronoi diagram. In this paper, we introduce a new algorithm [2] for the dynamic construction and maintenance of additively weighted Voronoi diagrams. We will illustrate the link of our algorithm with the Appolonius problem, and we will use some of the properties of hyperbolic Dirichlet tessellations.

2. Preliminaries

Let $\mathbb{N}$ be the set of integers, $\mathbb{R}$ be the set of reals, and $\mathbb{R}^2$ be the euclidean plane. Let $\mathcal{P}$ be the set of generators or sites. Let w_i be the weight of the point $P_i \in \mathcal{P}$. The Voronoi region of P_i is defined by:
$\mathcal{V}(P_i) = \left\{ M \epsilon \mathbb{R}^2 \ / \forall j : d(M, P_i) - w_i \leq d(M, P_j) - w_j \right\}$.

The additively weighted Voronoi diagram for the set of points $\mathcal{P}$ is defined as $\mathcal{V}(\mathcal{P}) = \{\mathcal{V}(P_i) / P_i \in \mathcal{P}\}$ (see Figure 1).

The bisector $\mathcal{B}_{ij}$ between P_i and P_j is defined by:
$\mathcal{B}_{ij} = \left\{ M \epsilon \mathbb{R}^2 \ / \forall j : d(M, P_i) - w_i = d(M, P_j) - w_j \right\}$.

The valence of a vertex of a tessellation is the number of regions to which it belongs (from section 2 of [3]).

A tessellation is proper if each region is regular closed and at each vertex the angles formed by the tangent rays to the boundary curves which meet there are all less than π (see section 2 of [3]).

Let $\mathcal{C}(P, w)$ be the weight circle of P, whose centre is P and radius is w.

It is evident that $\mathcal{B}_{ij} = \mathcal{V}(P_i) \cap \mathcal{V}(P_j)$. We can express $\mathcal{B}_{ij}$ in the following way:
$\mathcal{B}_{ij} = \left\{ M \epsilon \mathbb{R}^2 \ / \forall j : d(M, P_i) - d(M, P_j) = w_i - w_j \right\}$.

This locus is the branch of hyperbola whose foci are P_i and P_j, that is convex if $w_i \leq w_j$ (see Figure 1), and whose big axis is $a = \frac{w_i - w_j}{2}$.

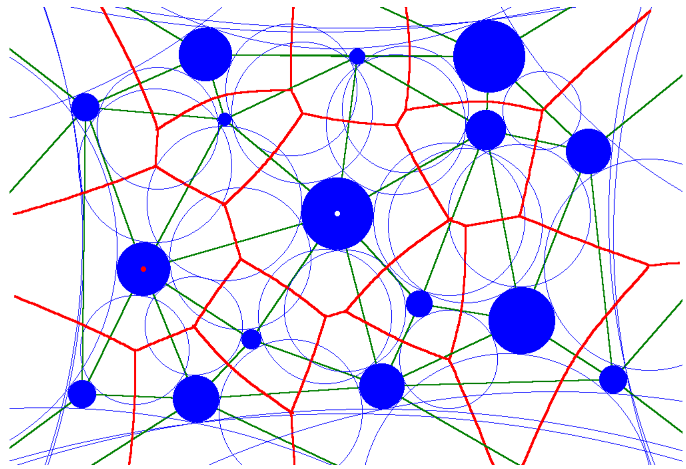

Figure 1: An additively Weighted Voronoi diagram.

The equation of such a hyperbola in polar coordinates is: $\rho = \frac{p}{1 - e \cos \theta}$ where e is the excentricity of the hyperbola: $e = \frac{d(P_i, P_j)}{|w_i - w_j|}$, and $p = a\left(e^2 - 1\right)$. Each branch of the hyperbola is obtained by adding the supplementary condition: $\rho \geq 0$ or $\rho \leq 0$.

From the equation of the hyperbola, it is trivial that any ray from P_i intersects at most once each branch of hyperbola. Therefore, $\mathcal{B}_{ij}$ is star-shaped relatively to P_i, and $\mathcal{V}(P_i)$ is star-shaped relatively to P_i.

According to lemma 3 of [3], $\mathcal{V}(P_i)$ is not empty if, and only if, $|w_i - w_j| < d(P_i, P_j)$.

Ash and Bolker proved that hyperbolic Dirichlet tessellations whose vertecies are 3-valent are proper (see Theorem 19 of [3]).

The intersection of $\mathcal{B}_{ij}$ and $\mathcal{B}_{ik}$ is the locus of the points M of the plane such as $d(M, P_i) - w_i = d(M, P_j) - w_j = d(M, P_k) - w_k$. This locus corresponds to the centre of the circle externally tangent to the circle's centre P_i radius w_i, centre P_j radius w_j, and centre P_k radius w_k (shown as thick plain circles on Figure 2). It is a solution of the Appolonius problem (see section 10.11.1 of [4] and Figure 2). It is also the intersection of $\mathcal{B}_{ij}$ and $\mathcal{B}_{jk}$ and of $\mathcal{B}_{ik}$ and $\mathcal{B}_{jk}$.

3. The algorithm

We will suppose that we do not allow $|w_i - w_j| \geq d(P_i, P_j)$ for any P_i, and P_j of $\mathcal{P}$. Therefore, every vertex is 3-valent, and the tesselation is proper. Therefore, the dual

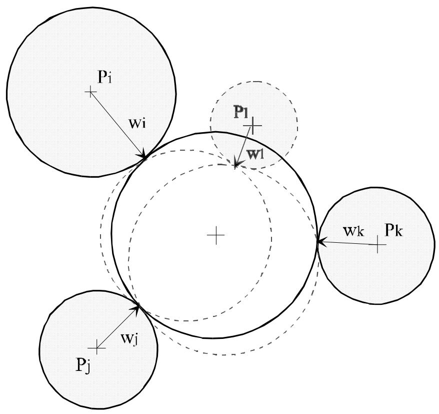

Figure 2: The event that changes the topology

graph of the additively weighted Voronoi diagram is a triangulation. Now, we will examine the events that affect this triangulation.

Proposition 1 *(The empty circumcircle criterion for the AW-Voronoi's dual graph): A triangle* $P_i\overset{\triangle}{P_j}P_k$ *exists in the triangulation if, and only if, the circle tangent to the weight circles* $\mathcal{C}(P_i, w_i)$, $\mathcal{C}(P_j, w_j)$, *and* $\mathcal{C}(P_k, w_k)$, *does not intersect any other circle* $\mathcal{C}(P_l, w_l)$ $l \notin \{i, j, k\}$.

Proof: If a fourth circle $\mathcal{C}(P_l, w_l)$ happened to be tangent to the circle $\mathcal{C}_{t_{\{i,j,k\}}}$, that is tangent to $\mathcal{C}(P_i, w_i)$, $\mathcal{C}(P_j, w_j)$, and $\mathcal{C}(P_k, w_k)$, then the vertex $v_{\{i,j,k\}}$ intersection of $\mathcal{B}_{ij}$, $\mathcal{B}_{ik}$, and $\mathcal{B}_{jk}$ would be 4-valent. This is not possible according to the assumption stated at the begining of this section. Otherwise, if the intersection of $\mathcal{C}(P_l, w_l)$ and $\mathcal{C}_{t_{\{ijk\}}}$ was constituted by two different points, then $\mathcal{C}_{t_{\{i,j,l\}}}$ and $\mathcal{C}_{t_{\{j,k,l\}}}$ would be tangent to $\mathcal{C}(P_i, w_i)$, $\mathcal{C}(P_j, w_j)$, and $\mathcal{C}(P_l, w_l)$; and $\mathcal{C}(P_j, w_j)$, $\mathcal{C}(P_k, w_k)$, and $\mathcal{C}(P_l, w_l)$ respectively. Then we would have the triangles $P_i\overset{\triangle}{P_j}P_k$, $P_i\overset{\triangle}{P_j}P_l$, and $P_j\overset{\triangle}{P_k}P_l$, which would contradict the fact that the dual graph of the additively weighted Voronoi diagram is a triangulation (see Figure 2). We should therefore make a triangle switch: replace $P_i\overset{\triangle}{P_j}P_k$ and $P_i\overset{\triangle}{P_k}P_l$ by $P_i\overset{\triangle}{P_j}P_l$ and $P_j\overset{\triangle}{P_k}P_l$.

This proposition is the basis of the incremental algorithm, that we implemented for the dynamic construction and maintenance of additively weighted Voronoi diagrams.

When a new point is added, we locate the triangle T in which it lies, then we connect this new point to the triangulation by replacing T by three new triangles whose vertices are the vertices of T and the new point. Then we check every circle tangent to the weight circles of the points of every new triangle. If a triangle switch has to be performed (see end of the Proof of Proposition 1), we perform the same check for all the tangent circles corresponding to the triangles generated by the triangle switch. This follows the algorithm of Guibas and Stolfi [5] for the ordinary Voronoi diagram, extending it to this case of generalized Dirichlet tessellation.

When an existing point is deleted, we locate its nearest neighbour, then we transfer all its neighbours to the nearest neighbour and we remove it and its topological relationships from the triangulation. Then we check every circle tangent to the weight circles of the points of every modified triangle. If a triangle switch has to be performed (see end of the Proof of Proposition 1), we perform the same check for all the tangent circles corresponding to the triangles generated by the triangle switch.

We prove that this algorithm has a worst case expected efficency of $O(\log n)$ per operation. This is due to the triangulation check, and the fact that each point has in average at most six neighbours.

4. Conclusion

This algorithm can be of interest for determining the deformation of materials [7], in crystallography and thermodynamics. Our algorithm can be useful in the applications of the additively weighted Voronoi diagrams or hyperbolic Dirichlet tessellations to spatial analysis (such as determining optimal investment location relatively to costs and aids from governments) due to its dynamic nature.

Acknowledgments

The second and third authors have received the financial support from a grant of the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Association des Industries Forestières du Québec (AIFQ).

REFERENCES

[1] F. Aurenhammer, Voronoi diagrams - A survey, (Institute for Information Processing, Technical University of Graz, Report 263).

[2] F. Aurenhammer, Personal communication during his visit to our research chair at Laval University, 1997.

[3] P.F. Ash, and E.D. Bolker, Generalized Dirichlet Tessellations, Geometria Dedicata No 20, 209-243.

[4] M. Berger, Géométrie, volume 2 : espaces euclidiens, triangles, cercles et sphères, (CEDIC/FERNAND NATHAN, Paris, 1979) 216p.

[5] Guibas, L., and Stolfi, J., 1985, Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams, ACM Transactions on Graphics, Vol. 4, No. 2, pp. 74-123.

[6] A. Okabe, B. Boots, and K. Sugihara, Spatial Tessellations, Concept and Applications of Voronoi Diagrams, (John Wiley & Sons, 1992) 532 p.

[7] F. Weller, Stability of Voronoi Neighborship under Perturbations of the Sites, (CCCG'97, Kingston, Canada, 1997) 251-256.