

An algorithm for the dynamic construction and maintenance of Additively Weighted Voronoi diagrams

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1. Introduction

The ordinary point Voronoi diagram is a partition of the plane, in the way that each object (point) partitions the euclidean plane into a region, that is the locus of points which are closer from that object than from any other object (see Preparata and Shamos [7]). The Voronoi diagram has many applications in a variety of disciplines, and has been widely treated in the literature (see Okabe and *al.* [6] and Aurenhammer [1] for a general survey).

The weighted Voronoi diagrams (multiplicatively, additively, compoundly, etc...) differ from the ordinary Voronoi diagram in the fact that the generators do not have all the same weight (see Okabe and *al.* [6] and Aurenhammer [1]). The definition of these weighted Voronoi diagrams differs from the definition of the ordinary one in the fact that the Euclidean distance is replaced by a weighted distance. In the case of the additively weighted Voronoi diagram, the weighted distance between a point and a generator, is the Euclidean distance minus the weight of the generator. The additively weighted Voronoi diagram has been extensively studied by Ash and Bolker, under the name of hyperbolic Dirichlet tessellations (see section 2 of [3]). Up to date, only static algorithms have been developed for constructing the additively weighted Voronoi diagram. In this paper, we introduce a new algorithm (see Aurenhammer [2]) for the dynamic construction and maintenance of additively weighted Voronoi diagrams. We will illustrate the link of our algorithm with the Apollonius problem, and we will use some of the properties of the hyperbolic Dirichlet tessellations.

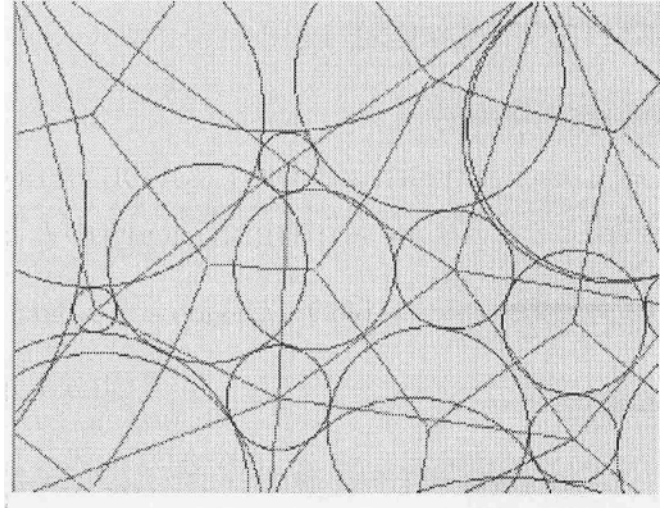


Figure 1: The Additively Weighted Voronoi diagram of a set of points (hyperbolas), its dual graph (edges), the weight circles, and the tangent circles

2. Preliminaries

Let \mathbb{N} be the set of integers, \mathfrak{R} be the set of reals, and \mathfrak{R}^2 be the euclidean plane. Let P be the set of generators or sites. Let w_i be the weight of the point $P_i \in P$. The Voronoi region of P , is defined by:

$$v(P_i) = \{M \in \mathfrak{R}^2 / \forall j : d(M, P_i) - w_i \leq d(M, P_j) - w_j\}$$

The additively weighted Voronoi diagram for a set of points P is defined as $v(P) = \{v(P_i) / P_i \in P\}$ (see figure 1).

The bisector B_{ij} between P_i and P_j is defined by

$$B_{ij} = \{M \in \mathfrak{R}^2 / \forall j : d(M, P_i) - w_i \leq d(M, P_j) - w_j\}$$

The valence of a vertex of a tessellation is the number

of regions to which it belongs (from section 2 of Ash and Bolker [3]).

A tessellation is proper if each region is regular closed and at each vertex the angles formed by the tangent rays to the boundary curves which meet there are all less than π (see section 2 of Ash and Bolker [3]).

Let $C(P, w)$ be the weight circle of P , the circle whose centre is M and radius is r .

It is evident that $B_{ij} = v(P_i) \cap v(P_j)$. We can express B_{ij} in the following way: $B_{ij} = \{M \in \mathbb{R}^2 / \forall j : d(M, P_i) - d(M, P_j) = w_i - w_j\}$.

This locus is the branch of hyperbola whose foci are P_i and P_j , that is convex if $w_i \leq w_j$ (see figure 1), and

$$\text{whose big axis is } a = \frac{w_i - w_j}{2}$$

The equation of such hyperbola in polar coordinates is:

$$p \frac{1}{1 - e \cos \theta} \text{ where } e \text{ is the excentricity of the hyperbola: } e = \frac{d(P_i, P_j)}{|w_i - w_j|}, p = a(e^2 - 1)$$

the axis of the polar system is the edge (P_i, P_j) and its pole is P_i . Each branch of hyperbola is obtained by adding the supplementary condition: $p \geq 0$ or $p \leq 0$.

From the equation of the hyperbola, it is trivial that any ray from P_i intersects at most once each branch of hyperbola. Therefore, B_{ij} is starshaped relatively to P_i and $V(P_i)$ is starshaped relatively to P_i .

According to lemma 3 of Ash and Bolker [3], $V(P_i)$ is not empty if, and only if, $|w_i - w_j| < d(P_i, P_j)$.

Ash and Bolker proved that hyperbolic Dirichlet tessellations whose vertecies are 3-valent are proper (see theorem 19 of [3]).

The intersection of B_{ij} and B_{ik} is the locus of the points M of the plane such as $d(M, P_i) - w_i = d(M, P_j) - w_j = d(M, P_k) - w_k$. This locus corresponds to the centre(s) of the circle(s) externally tangent to the circles centre P_i radius w_i centre P_j radius w_j and centre P_k radius w_k . It is a solution of the Appolonius problem (see section 10.11.1 of Berger [4] and figure 2). It is also the intersection of B_{ij} and B_{ik} and of B_{ik} and B_{jk} .

3. The algorithm

We will suppose that we do not allow $|w_i - w_j| \geq d(P_i, P_j)$ for any P_i, P_j of P . Therefore, every vertex is 3-valent, and the tessellation is proper. Therefore, the dual graph of the additively weighted Voronoi diagram is a triangulation. Now, we will examine the events that affect this triangulation (also called topological events by [5]).

Proposition 1 A triangle $P_i P_j P_k$ exists in the triangulation if, and only if, the circle tangent to the weight circles $C(P_i, w_i)$, $C(P_j, w_j)$, and $C(P_k, w_k)$, does not intersect any other circle $C(P_l, w_l) / l \notin \{i, j, k\}$.

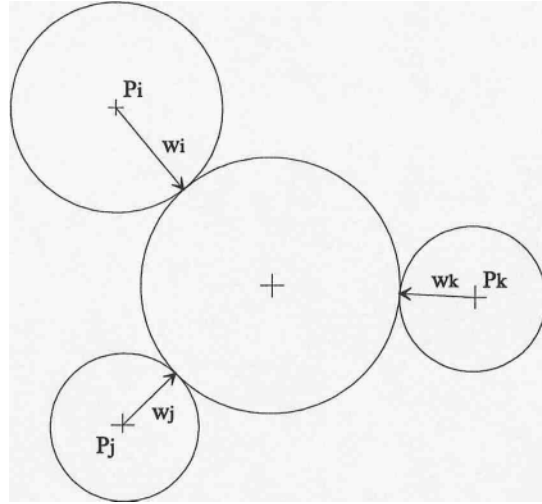


Figure 2: The additively weighted Voronoi diagram vertex: a particular case of the Appolonius problem.

Proof: If a fourth circle $C(P_l, w_l)$ happened to be tangent to the circle $C_{t\{i,j,k\}}$ that is tangent to $C\{P_i, w_i\}$, $C\{P_j, w_j\}$, and $C\{P_k, w_k\}$, then the vertex $v_{\{i,j,k\}}$ intersection of B_{ij} , B_{ik} , and B_{jk} would be 4-valent. This is not possible according to the assumption stated at the beginning of this section. Otherwise, if the intersection of $C(P_l, w_l)$ and $C_{t\{i,j,k\}}$ was constituted by two different points, then $C_{t\{i,j,l\}}$ and $C_{t\{j,k,l\}}$ would be tangent to $C\{P_i, w_i\}$, $C\{P_j, w_j\}$, and $C\{P_k, w_k\}$, and $C\{P_l, w_l\}$ respectively. Then we would have the triangles $P_i P_j P_k$ and $P_i P_j P_l$ and $P_j P_k P_l$ what would be in contradiction with the fact that the dual graph of the additively weighted Voronoi is a triangulation (see figure 3). We should therefore make a triangle switch: replace $P_i P_j P_k$ and $P_i P_k P_l$ and $P_i P_j P_l$ and $P_j P_k P_l$.

This proposition is the basis of the incremental algorithm, that we implemented for the dynamic construction and maintenance of additively weighted Voronoi diagrams.

When a new point is added, we locate the triangle T in which it lies, then we connect this new point to the triangulation by replacing T by three new triangles whose vertices are the vertices of T and the new point. Then we check every circle tangent to the weight circles of the points of every new triangle. If a topological event occurs, we perform the same check for all the tangent circles corresponding to the triangles generated by the triangle switch.

When an existing point is deleted, we locate its nearest neighbour, then we transfer all its neighbours to the nearest neighbour and we remove it and its topologi-

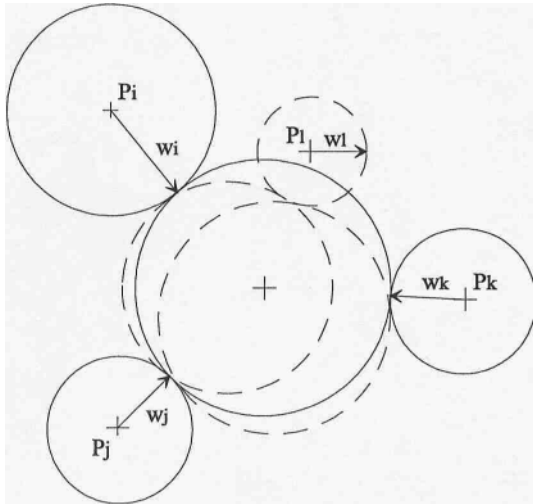


Figure 3: The event that changes the topology

cal relationships from the triangulation. Then we check every circle tangent to the weight circles of the points of every modified triangle. If a topological event occurs, we perform the same check for all the tangent circles corresponding to the triangles generated by the triangle switch.

We prove that this algorithm has a worst case expected efficiency of $\theta(\log n)$ per operation. This is due to the triangulation check, and the fact that each point has in average at most six neighbours.

4. Conclusion

This algorithm can be of interest in the determination of the "Stability of Voronoi Neighborhood under Perturbations of the Sites" (see Weller [8]), due to the fact that this determination uses additively weighted Voronoi diagrams. It is of interest for determination of deformation of materials, in crystallography and thermodynamics. Our algorithm can be very useful in the applications of the additively weighted Voronoi diagrams or hyperbolic Dirichlet tessellations to spatial analysis (such as determining optimal investment location relatively to costs and aids from governments) due to its dynamic nature. Finally, we are working on the implementation of the kinematic version of our algorithm.

References

[1] F. Aurenhammer, Voronoi diagrams - A survey, (Institute for Information Processing, Technical University of Graz, Report

- [2] F. Aurenhammer, Personal communication during his visit to our research chair at Laval University, 1997.
- [3] P.F. Ash, and E.D. Bolker, Generalized Dirichlet Tessellations, *Geometria Dedicata* No 20, 209-243
- [4] M. Berger, *Geometric*, volume 2 : espaces euclidiens, triangles, cercles et spheres, (CEDIC/FERNAND NATHAN, Paris, 1979) 216p.
- [5] C.M. Gold, P.R. Remmele, and T. Roos, *Voronoi Diagrams of Line Segments Made Easy*, (CCCG'95, Quebec, Canada, 1995) 223-228.
- [6] A. Okabe, B. Boots, and K. Sugihara, *Spatial Tessellations, Concept and Applications of Voronoi Diagrams*, (John Wiley & Sons, 1992) 532 p.
- [7] P.P. Preparata, and M.I. Shamos, *Computational Geometry, An Introduction*, (Springer-Verlag, 1985) 398 p.
- [8] F. Weller, *Stability of Voronoi Neighborhood under Perturbations of the Sites*, (CCCG'97, Kingston, Canada, 1997) 251-256