AN INCREMENTAL ALGORITHM FOR THE COMPUTATION OF PLANAR JOHNSON-MEHL TESSELLATIONS

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1. Introduction

The Johnson-Mehl model has been introduced in [3] for modelling the growth of crystal aggregates. The Johnson-Mehl model is a Poisson Voronoi growth model, in which nuclei are generated asynchronously using a Poisson point process [5], and grow at the same radial speed v. Each generator P_i has both a planar location (its position vector) and an associated birth time t_i ($t_i \geq 0$). The Johnson-Mehl tessellation can be considered equivalent to a dynamic version of an additively weighted Voronoi diagram, in which the weight reflects the arrival time of the point in $\mathbb{R}^2$ [5]. The additively weighted Voronoi diagram has been extensively studied by Ash and Bolker, under the name of hyperbolic Dirichlet tessellations [2], but till [4] and [1], there was no dynamic algorithm for constructing the additively weighted Voronoi diagram. In this paper, we introduce a new algorithm based on an algorithm for dynamic construction of additively weighted Voronoi diagrams [4], [1] for the incremental construction and maintenance of Johnson-Mehl tessellations.

2. Preliminaries

Let $\mathbb{R}^2$ be the euclidean plane. Let $\mathcal{P} = \{P_i\}$ be the set of generators or nuclei, where $P_i = (\overrightarrow{p_i}, t_i)$ is defined by its position vector $\overrightarrow{p_i}$, and its birth time t_i. The cell $\mathcal{C}(P_i)$ generated by the growth of P_i contains all the points M such that the cell growing from P_i is the first one to meet M.

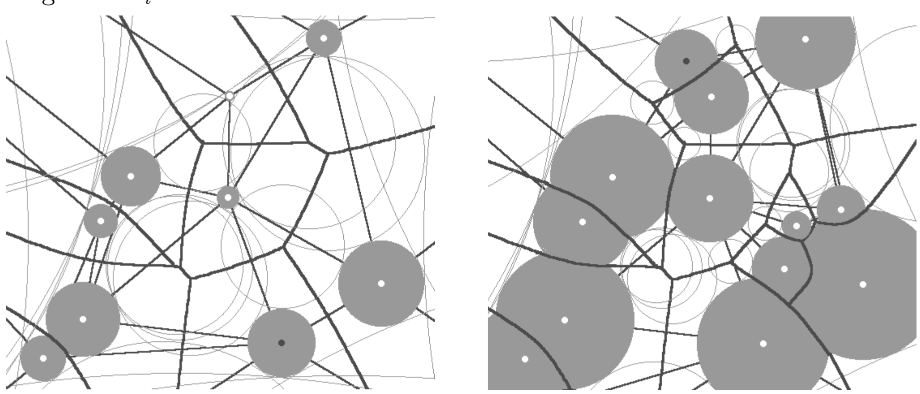

Figure 1: The Johnson-Mehl tessellation at two different times.

It is defined by: $\mathcal{C}(P_i) = \left\{ M\epsilon\mathbb{R}^2 \,/\forall j : t_i + \frac{d(M,P_i)}{v} \leq t_j + \frac{d(M,P_j)}{v} \right\}$, where v is the constant radial growth of the nuclei. If we multiply both sides of the inequality inside the brackets by $v \geq 0$, and substract vt_c where t_c is the current time, we get:

$\mathcal{C}(P_i) = \left\{ M \epsilon \mathbb{R}^2 \,/ \forall j : d(M, P_i) - v(t_c - t_i) \leq d(M, P_j) - v(t_c - t_j) \right\}$. This establishes the correspondence between the Johnson-Mehl tessellation and the additively weighted Voronoi diagram mentioned in the previous section. The Johnson-Mehl tessellation for the set of points $\mathcal{P}$ is defined as $\mathcal{C}(\mathcal{P}) = \{\mathcal{C}(P_i) \,/ P_i \in \mathcal{P}\}$ (see Figure 1). The cell $\mathcal{C}(P_i)$ is not empty if and only if for all j, $|v(t_j - t_i)| < d(M, P_j)$, that is to say the weight circle of P_i is not inside the weight circle of P_j.

3. The algorithm

The nuclei are generated by a homogeneous Poisson point process. We recall now a condition of the Johnson-Mehl model. If a point is born at a location that is already occupied by a growing cell, it disappears without trace [5]. This implies that no cell is empty in $\mathcal{C}(\mathcal{P})$, every vertex is 3-valent, and the Johnson-Mehl tessellation is proper. The dual graph of the additively weighted Voronoi diagram is a triangulation that obeys the Delaunay triangulation "empty circumcircle criterion" (Proposition 1 of [4]). This is the basis of the incremental algorithm we implemented for the construction and maintenance of the Johnson-Mehl model. After each arrival of a new nucleus, the Johnson-Mehl tessellation changes, and we recompute it as in the algorithm presented in [4]. This algorithm is also applied in the case of a time inhomogeneous Poisson point process when all the nuclei grow at the same radial speed for each time interval. Consequently, as long as a new nucleus does not arrive, the difference between the weights of neighbouring nuclei is constant, and the Johnson-Mehl tessellation does not change. Finally, this dynamic algorithm can be of interest in crystallography, material science and in the applications of the Johnson-Mehl tessellations to spatial analysis.

References

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